2. Mathematics is model-building of the real world. Mathematics is about solving problems from the real world and generalising the results so that they can be used on similar problems. A mathematical model or theory then consists of some mathematical objects and some rules valid for these objects. However, these objects are not created in an arbitrary way, they are inspired by the real world. Once a mathematical model has been created, this model can in itself include problems, which will create a new mathematical model. Mathematics is learned, because it is needed to solve certain problems in the real world.

Several things indicate that quite a number of pupils and teachers have the first perception. Is this desirable, and why is it so? Is it because mathematics is presented in an isolated and deductive way? On the other hand, mathematics presented together with induction and applications, is that real mathematics? Are the applications only a spoonful of sugar, to help the medicine go down? Or do they constitute the reason and explanation for taking the medicine?

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A construction for a regular heptagon

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Call the ends of a ruler A and Z, and (towards A) place on it a mark X. Draw a line BC the length of AX, and a square $BCDE$. Erect a perpendicular bisector of BC. With centre C and radius CE draw a circle. Now place the ruler (Fig. 1) so that AZ passes through B, with A on the perpendicular bisector of BC, and with X on the circle. Then $\angle BAC = \pi/7$.

PROOF. Take $BC = AX$ to be of unit length and let $AC = x$, $\angle BAC = 2\theta$ (Fig. 2). Then, from triangle ABC,

$$2x \sin\theta = 1;$$

and, from triangle AXC, the cosine formula gives

$$2 = 1 + x^2 - 2x\cos 2\theta.$$

Eliminating x leads to

$$8\sin^3\theta - 4\sin^2\theta - 4\sin\theta + 1 = 0,$$

which is the same as the equation we obtain by expressing the relation

$$\sin 3\theta = \cos 4\theta$$

in terms of $\sin\theta$ and removing the factor $\sin\theta + 1$. Its roots are therefore

$$\theta = \frac{\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \ldots$$

so that, in Fig. 1, $\angle BAC = \pi/7$. (We leave it to the reader to find the constructions which lead to the roots other than $\pi/14$.)

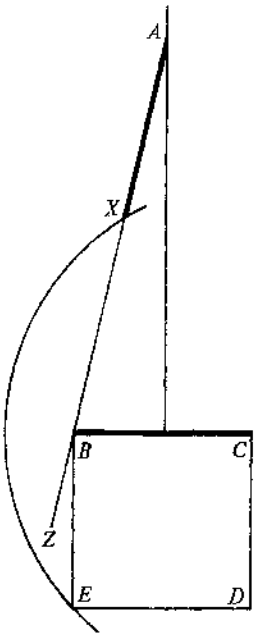

FIGURE 1.

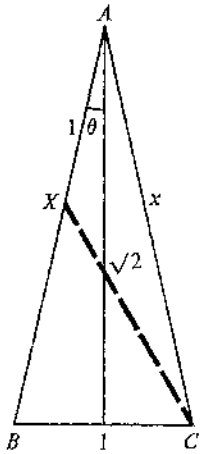

FIGURE 2.

A one-mark construction of the heptagon is apt to lend credence to a legend of a "lost neusis" of Archimedes. In a summary of knowledge of the neglected heptagon in *Mathematics Magazine* (**46** (No. 1), 1973) Bankoff and Garfunkel say: "According to Arabian sources, Archimedes is believed to have written a book on the heptagon inscribed in a circle. If it is true that this work ever existed, it now seems to be irretrievably lost. Still, the question of its having been written appears credible because of a single surviving proposition, namely a 'neusis' or 'verging' construction of a regular heptagon. Archimedes accomplished this brilliant feat by using a marked instead of an unmarked ruler and by placing a certain line segment of definite length at a specially manipulated position in relation to certain other points and lines. Details elucidating this vague description may be found in Heath's *Manual of greek mathematics* on pages 340–2 of the Dover reprint."

The discussion by Heath is of a reconstruction by Thābit ben Qurra from an almost undecipherable Greek manuscript, now lost, that Thābit attests to have been the work of Archimedes, and that with astonishing geometric logic arrives at a correct construction of the heptagon. It does this by way of an auxiliary drawing (Fig. 3, without the broken lines) and

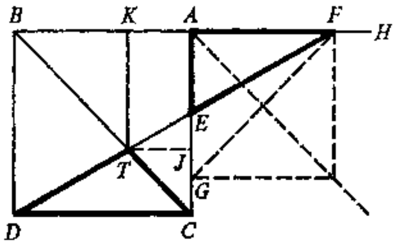

FIGURE 3.

proof that, when triangles EAF and DTC are equal in area, the transversal DF marks off at F on the horizontal one of the proportions of the heptagon.

It is of this drawing that Heath speaks. "Archimedes, in fact, according to our authority, reduced this auxiliary problem to a kind of neusis solved by means of a ruler, without troubling to show how it might alternatively be solved by means of conics or otherwise." And again he mentions the "neusis, which Archimedes apparently gave without any hint of how he arrived at it . . .".

It is inconceivable that anybody could find and use a one-mark solution to a problem anciently recognised as beyond euclidean procedure and not fully record it (as Newton recorded his doubled cube and as Archimedes himself recorded his trisection). Archimedes' progression from the auxiliary drawing to the drawing of the complete heptagon, for which no neusis is mentioned, also argues against his having had a true neusis. The neusis given here (Fig. 2) will of course construct both drawings, the complete heptagon more easily. A circle centred on X with radius AX will mark off a seventh of its circumference as it crosses lines AB and AC. Archimedes builds his heptagon with a triangle formed by the segments of line BF in the auxiliary construction (Fig. 3). Van der Waerden, in *Science awakening* (pp. 226–7), does not mention a neusis for either drawing. He crosses the gap in procedure in the auxiliary drawing by adding a conic bridge.

We still are left with Archimedes' words (in Heath's text) to account for. He does not say he had a neusis but a "kind of neusis", and as he describes the construction of the auxiliary drawing he certainly indicates he had something of the sort. After we are told to let $ABCD$ be a square with one side BA produced to H, we are instructed by Archimedes: "We draw the diagonal BC, then, bringing the ruler to point D, we direct the other end to point F on AH so chosen that (if DF meets the diagonal BC in T) the triangle FAE shall be equal to the triangle CTD." This is impossible to do unless the equality of the triangles brings about a recognisable linear condition. It does; the added broken lines (Fig. 3) will show TJ equal to EG when the triangles are equal and when a 45° diagonal from G indicates the 'chosen' point F. If AH has been drawn equal to BA then FH, EJ and GC are equal. Parallels and recurring squares offer a variety of guides from the $TJ:EG$ and $TJ:JC$ equalities to the dissimilar but equal triangles Archimedes uses in his proof.

This surely can be called a "kind of neusis" but the squares are not so obvious that it can go without explanation. Some of the words are lost, we may say; or we may conjecture that the interesting triangle DBF, with its squares and equal triangles, had a fuller existence before the discovery of its heptagon proportions. We know (from Van der Waerden) that the drawing is Proposition 16 (and the heptagon drawing Proposition 17) in a treatise that begins with a number of propositions on right triangles.

An observation (made too, incredibly, in Archimedes' Syracuse, in the fall of 1973) that the 1:3:3-angle triangle can be constructed by using seven toothpicks and the edges of a menu and wine list (Fig. 4) proved not to be original; the idea is to be found in a note, "Zig-zag paths", by Archibald H. Finlay, in *Gazette* **43**, 199 (No. 345, October 1959). However, the seven toothpicks are the key to the heptagon. The three segments of each side of the triangle (Fig. 4) correspond to the divisions of Archimedes' BF line (Fig. 3). If AB equals BF then AX is BA, XY is AF, YB is KA.

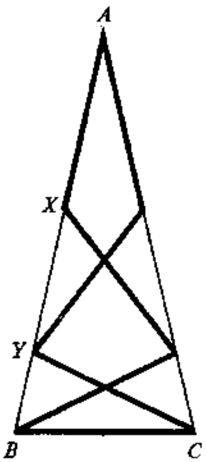

FIGURE 4.

The writer, a painter and not a mathematician, made use of the fascinatingly co-operative internal geometry of the polygon in constructions for a series of abstract paintings, the drawing for one of which led to the discovery of the $\sqrt{2}$ line and the neusis construction.

Postscript

After the completion of this article, the author received a letter from Mr. G. Stanley Smith of Seaford, Sussex (to whom he was already indebted for help with proofs), pointing out that, as with other neusis constructions (e.g. for trisecting an angle and duplicating the cube) the figure could conveniently be drawn with the aid of the conchoid having B as pole and the perpendicular bisector of BC as asymptote (see, for example, E. H. Lockwood's *Book of curves*, pp. 126–9). In Fig. 5 is shown the intersection

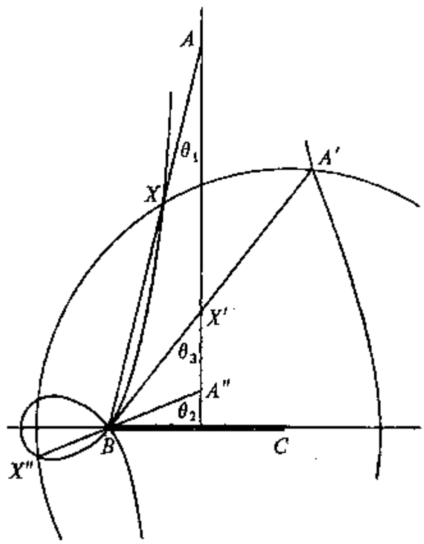

FIGURE 5.

of this conchoid with the circle centre C; θ_1 is the angle $\pi/14$ of the construction in Fig. 1, and θ_2, θ_3 are equal to $5\pi/14$ and $3\pi/14$ respectively. It will be seen that the latter angles also can be fixed by a second and third placing through B of the marked ruler, $A'X'$ and $A''X''$.

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